Lecture 6

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Example

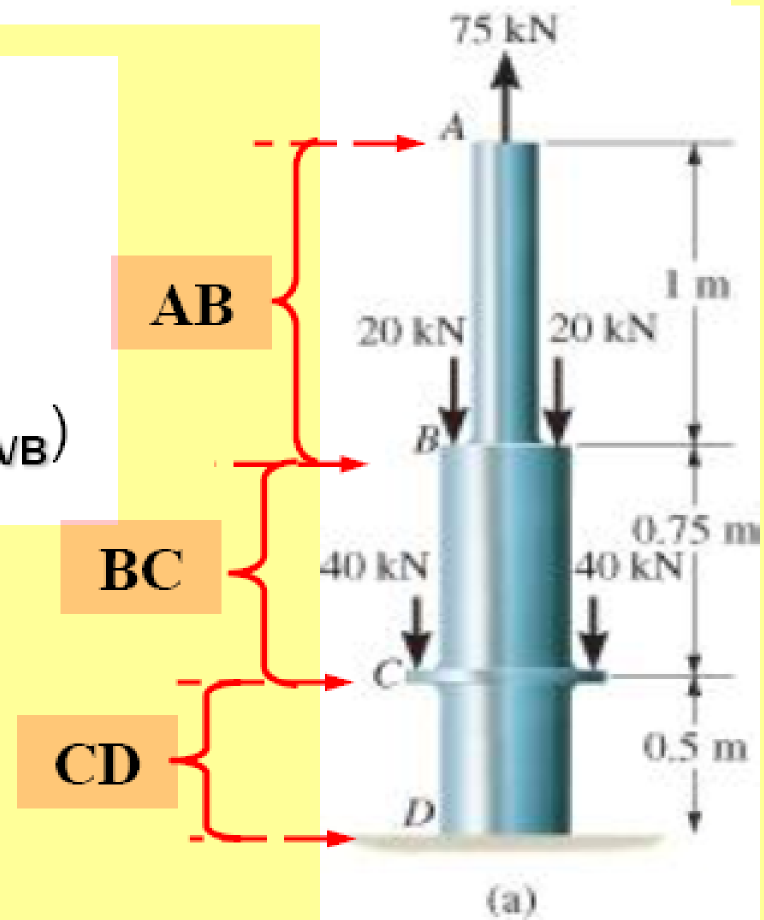
Composite A-36 steel bar shown made from two segments AB and BD.

Area A_{AB} = 600 mm^2 and A_{BD} = 1200 mm^2.

Determine the vertical **displacement of end A** (δ_A) and displacement of B relative to C ($\delta_{A/B}$)

.

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$



Example

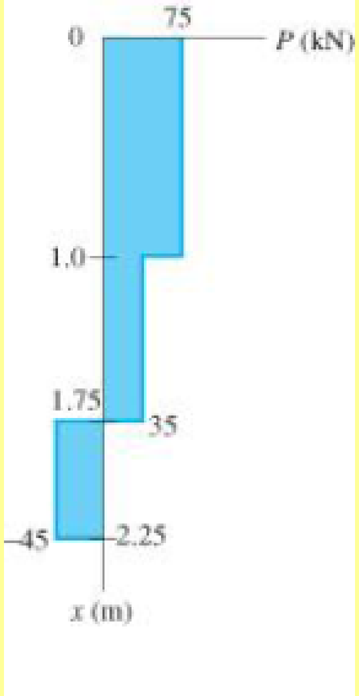
Solution:

Due to external loadings, internal axial forces in regions *AB*, *BC* and *CD* are different.

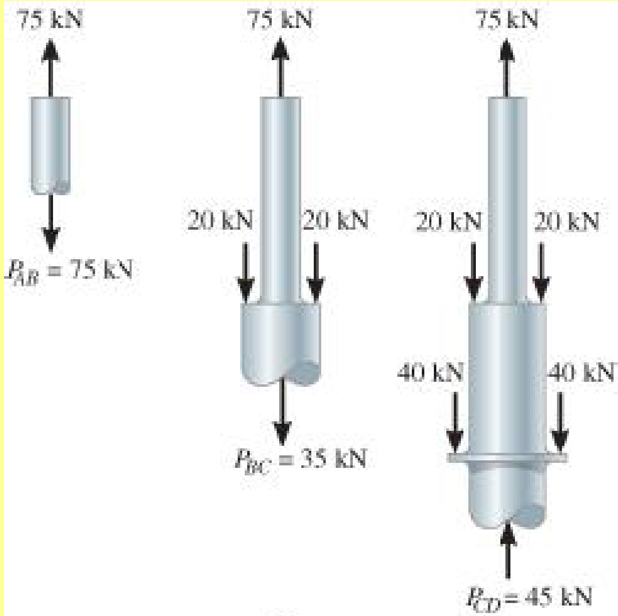
- **Find the Internal force in each portion by using section method and ΣFy=0 for vertical forces.**

- **Plot the variation of P.**

- **Plot the variation of σ**



(c)



(b)

Example

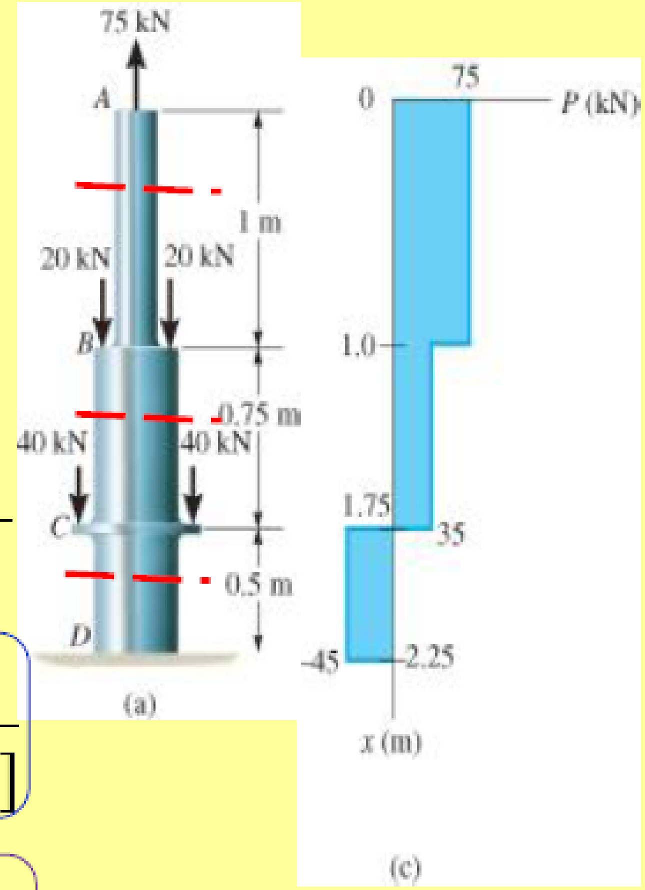
$E_{st} = 210(10^3)\ \text{kN/m}^2$ **(Given).**

Vertical displacement of A relative to fixed support D is:

$$\delta_A = \sum \frac{PL}{AE} =$$

$$\frac{[+75\ \text{kN}](1\text{m})(10^6)}{[600\ \text{mm}^2\ (210)(10^3)\ \text{kN/m}^2]} + \frac{[+35\ \text{kN}](0.75\ \text{m})(10^6)}{[1200\ \text{mm}^2\ (210)(10^3)\ \text{kN/m}^2]} + \frac{[-45\ \text{kN}](0.5\ \text{m})(10^6)}{[1200\ \text{mm}^2\ (210)(10^3)\ \text{kN/m}^2]}$$

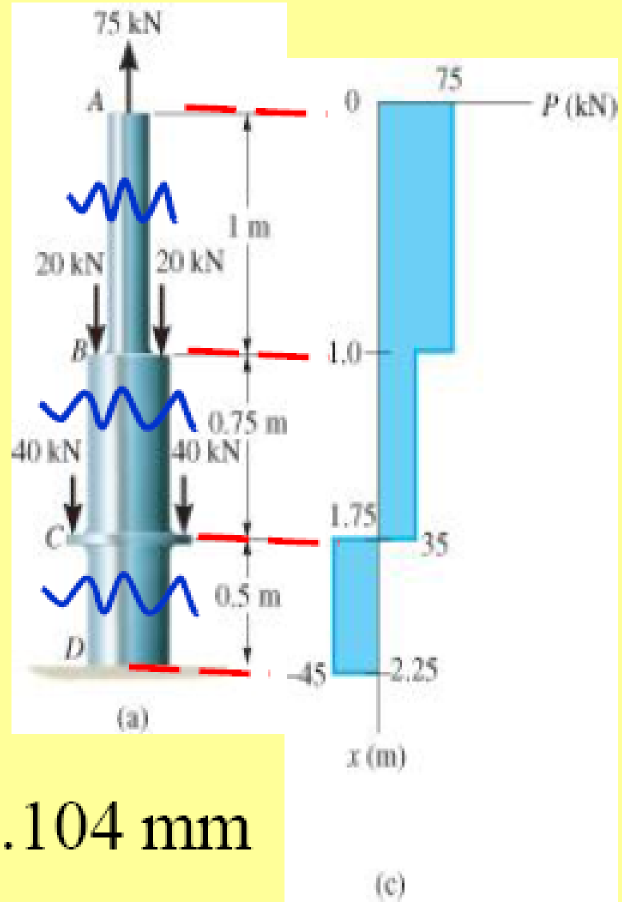
$= \mathbf{+0.61\ mm}$



Example

δ_A is +ve, so the bar elongates and so displacement at A is upward.

Find the displacement between B and C,

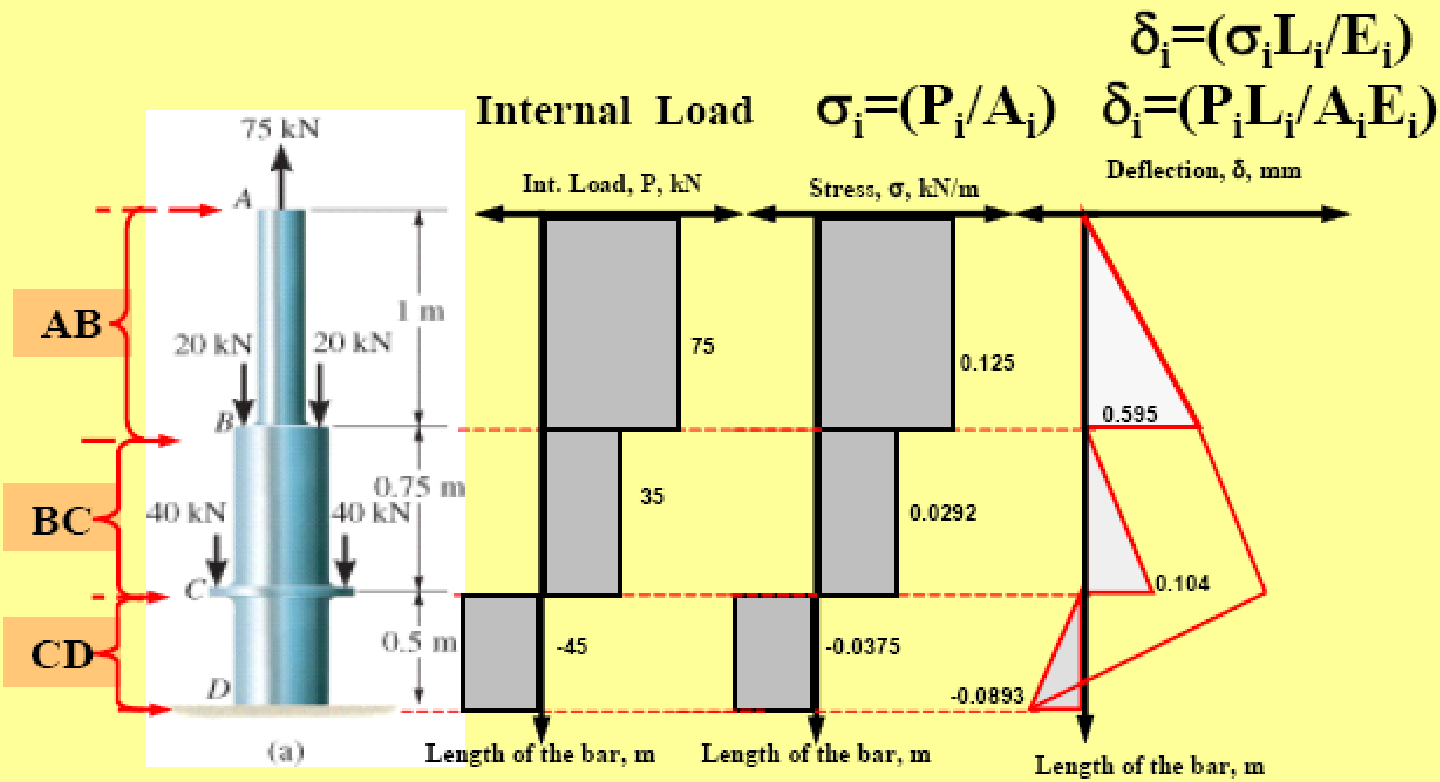


$$\delta_{B-C} = \frac{P_{BC} L_{BC}}{A_{BC} E} = $$

$$\frac{[+35 \text{ kN}](0.75 \text{ m})(10^6)}{[1200 \text{ mm}^2 \, (210)(10^3) \text{ kN/m}^2]} = +0.104 \text{ mm}$$

Here, B moves away from C, since segment elongates

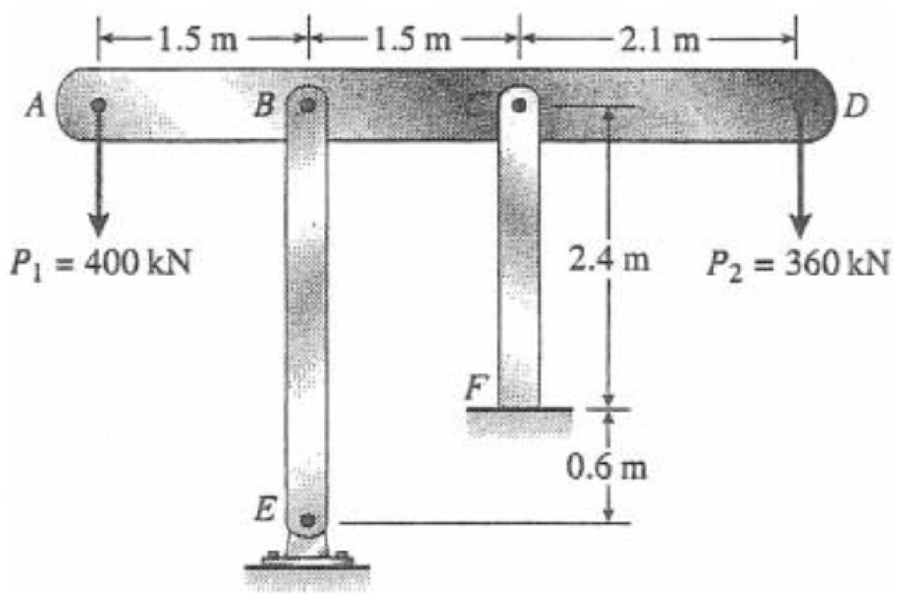
Example



Segment	P, kN	A, m^2	L, m	E, kN/m^2	σ, kN/m^2	δ, mm	Remarks
AB	75	600	1	$210x10^3$	0.125	0.595	Tension
BC	35	1200	0.75	$210x10^3$	0.0292	0.104	Tension
CD	-45	1200	0.5	$210x10^3$	-0.0375	-0.0893	Compression
						δ_t=+0.61	Point A moves upward

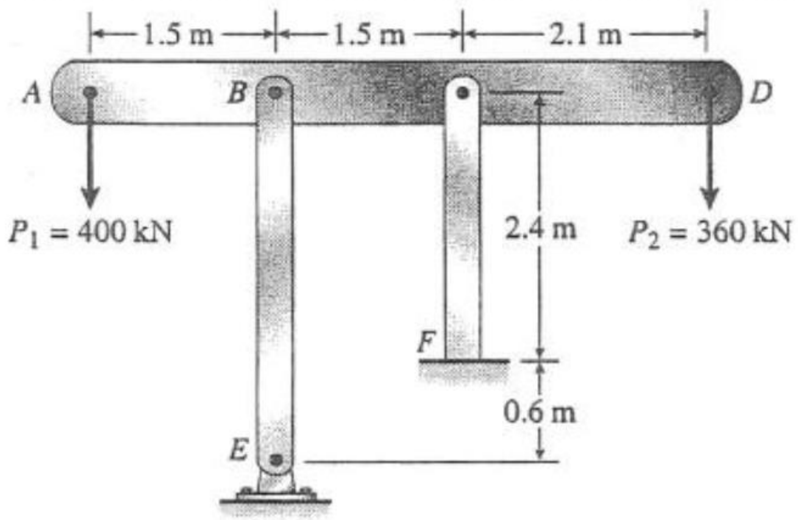
Example

The horizontal rigid beam *ABCD is supported* by vertical bars *BE and CF and is loaded by vertical forces* $P_1 = 400$ *kN and* $P_2 = 360$ *kN acting at points A and D,* respectively (see figure). Bars *BE and CF are made of steel* (E 200 GPa) *and have cross-sectional areas* $A_{BE} = 11{,}100\ mm^2$ and $A_{CF} = 9{,}280\ mm^2$. *The distances between various points on* the bars are shown in the figure. Determine the vertical displacements δ_A *and* δ_D *of points A* and *D, respectively.*



Example

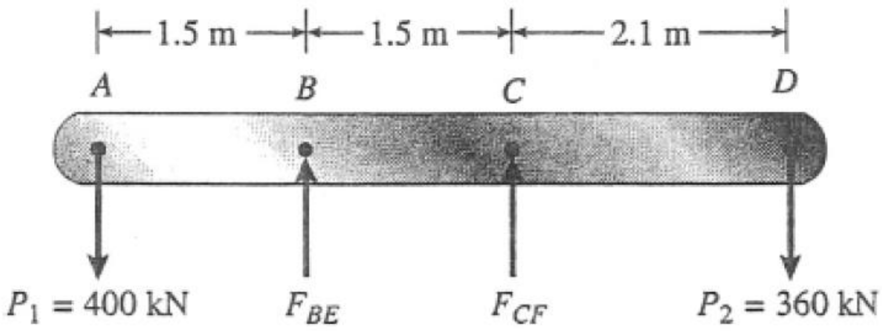
solution



$$A_{BE} = 11{,}100\ \text{mm}^2$$
$$A_{CF} = 9{,}280\ \text{mm}^2$$
$$E = 200\ \text{GPa}$$
$$L_{BE} = 3.0\ \text{m}$$
$$L_{CF} = 2.4\ \text{m}$$
$$P_1 = 400\ \text{kN};\ P_2 = 360\ \text{kN}$$

Example

FREE-BODY DIAGRAM OF BAR *ABCD*



$\Sigma M_B = 0$

$(400\text{ kN})(1.5\text{ m}) + F_{CF}(1.5\text{ m}) - (360\text{ kN})(3.6\text{ m}) = 0$

$F_{CF} = 464\text{ kN}$

$\Sigma M_C = 0$ 12

$(400\text{ kN})(3.0\text{ m}) - F_{BE}(1.5\text{ m}) - (360\text{ kN})(2.1\text{ m}) = 0$

$F_{BE} = 296\text{ kN}$

SHORTENING OF BAR *BE*

$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296\text{ kN})(3.0\text{ m})}{(200\text{ GPa})(11{,}100\text{ mm}^2)}$$

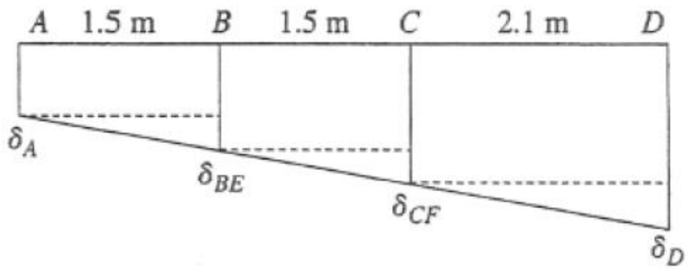
$$= 0.400\text{ mm}$$

Example

SHORTENING OF BAR CF

$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464\text{ kN})(2.4\text{ m})}{(200\text{ GPa})(9{,}280\text{ mm}^2)} = 0.600\text{ mm}$$

DISPLACEMENT DIAGRAM



$$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE} \text{ or } \delta_A = 2\delta_{BE} - \delta_{CF}$$

$$\delta_A = 2(0.400\text{ mm}) - 0.600\text{ m}$$
$$= 0.200\text{ mm} \leftarrow$$

(Downward)

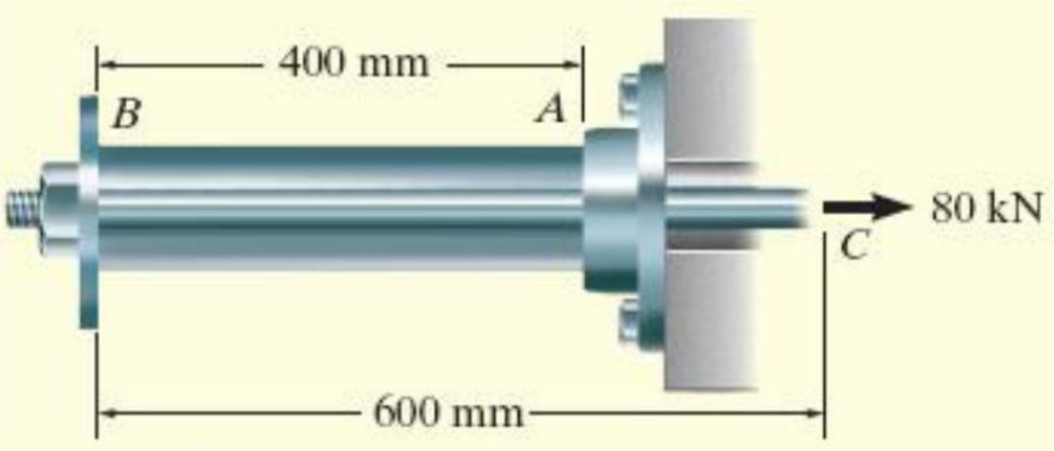
$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

or $$\delta_D = \frac{12}{5}\delta_{CF} - \frac{7}{5}\delta_{BE}$$
$$= \frac{12}{5}(0.600\text{ mm}) - \frac{7}{5}(0.400\text{ mm})$$
$$= 0.880\text{ mm} \leftarrow$$

(Downward)

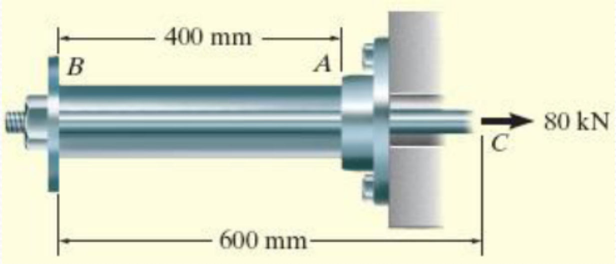
Example

The assembly consists of an aluminum tube *AB* having a cross-sectional area of 400 mm². A steel rod having a diameter of 10 mm is attached to a rigid collar and passes through the tube. If a tensile load of 80 kN is applied to the rod, determine the displacement of the end *C* of the rod. (E_{st} = 200 GPa, E_{al} = 70 GPa)



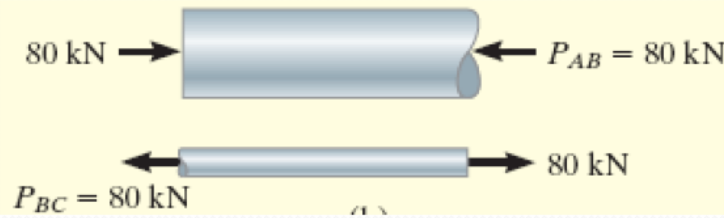
Example

Solution:



Find the displacement of end C with respect to end B.

$$\delta_{C/B} = \frac{PL}{AE} = \frac{\left[+80\left(10^3\right)\right](0.6)}{\pi(0.005)\left[200\left(10^9\right)\right]} = +0.003056\,\text{m} \rightarrow$$



Displacement of end B with respect to the *fixed* end A,

$$\delta_B = \frac{PL}{AE} = \frac{\left[-80\left(10^3\right)\right](0.4)}{\left[400\left(10^{-6}\right)\right]\left[70\left(10^9\right)\right]} = -0.001143 = 0.001143\,\text{m} \rightarrow$$

Since both displacements are to the right,

$$\delta_C = \delta_B + \delta_{C/B} = 0.0042\,\text{m} = 4.20\,\text{mm} \rightarrow$$

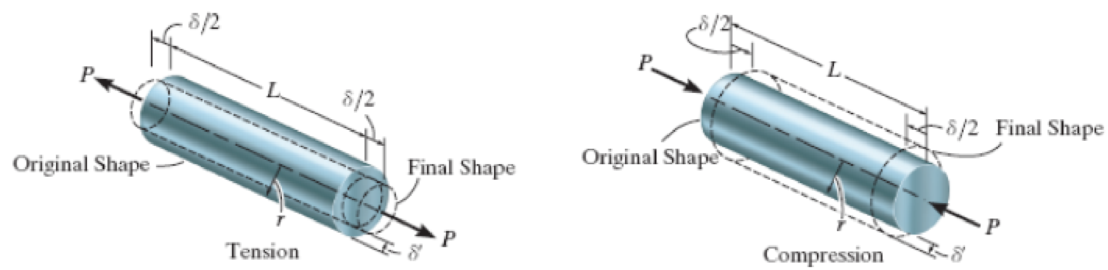
Poisson's ratio

- ***Poisson's ratio***, *ν* (nu), states that in the *elastic range*, the *ratio* of these strains is a *constant* since the deformations are proportional.

$$\nu = -\frac{\varepsilon_{lat}}{\varepsilon_{long}}$$

Poisson's ratio is *dimensionless*.
Typical values are 1/3 or 1/4.

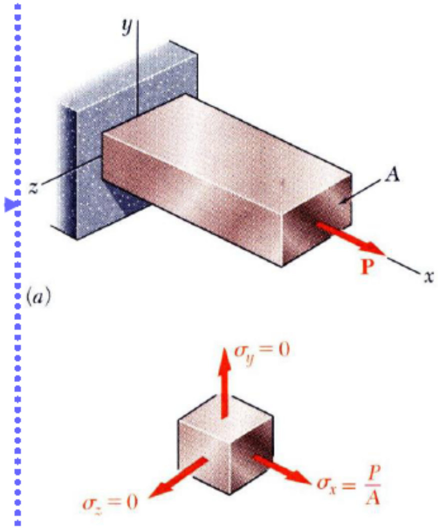
- Negative sign since *longitudinal elongation* (positive strain) causes *lateral contraction* (negative strain), and vice versa.



Poisson's ratio

- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$



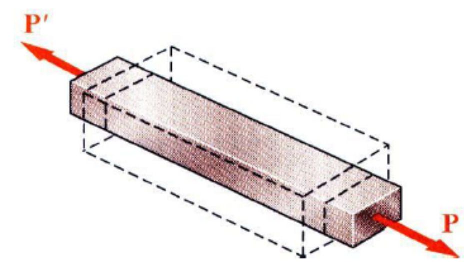
- The elongation in the *x*-direction is accompanied by a ***contraction*** in the other lateral directions. Assuming that the *material is isotropic* (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

- Poisson's ratio is defined as

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Note that ν is a property of the material.



- So; for a ***homogeneous isotropic*** material subjected to an ***axial loading in x*** direction:

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \text{and} \quad \varepsilon_y = \varepsilon_z = -\frac{\nu \sigma_x}{E}$$

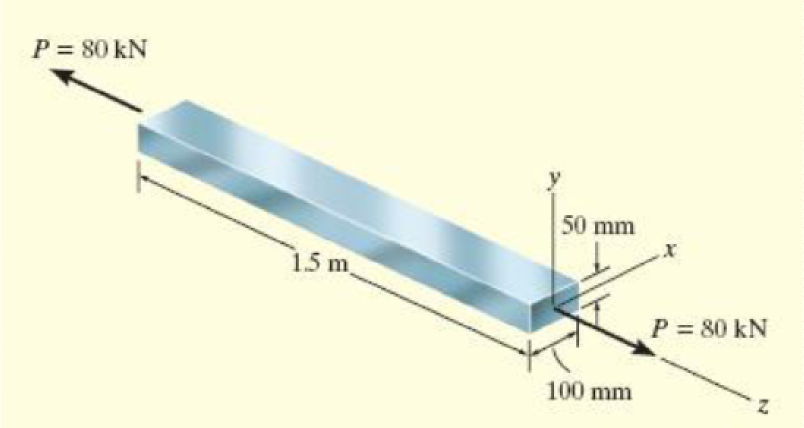
Example

A bar made of A-36 steel has the dimensions shown. If an axial force of is applied to the bar, determine the change in its length and the change in the dimensions of its cross section after applying the load. The material behaves elastically.

Solution:

The normal stress in the bar is

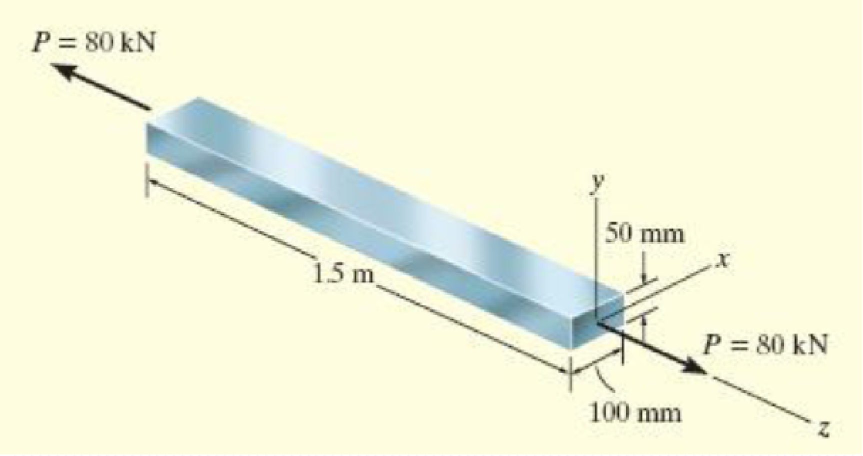
$$\sigma_z = \frac{P}{A} = \frac{80\left(10^3\right)}{(0.1)(0.05)} = 16.0\left(10^6\right)\text{Pa}$$



From the table for A-36 steel, E_{st} = 200 GPa

$$\varepsilon_z = \frac{\sigma_z}{E_{st}} = \frac{16.0\left(10^6\right)}{200\left(10^6\right)} = 80\left(10^{-6}\right)\text{mm/mm}$$

Example



Solution:

The axial elongation of the bar is therefore

$$\delta_z = \varepsilon_z L_z = \left[80\left(10^{-6}\right)(1.5)\right] = 120\,\mu\text{m} \text{ (Ans)}$$

The contraction strains in *both* the *x* and *y* directions are

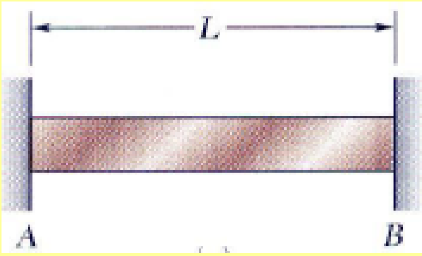
$$\varepsilon_x = \varepsilon_y = -\nu_{st}\varepsilon_z = -0.32\left[80\left(10^{-6}\right)\right] = -25.6\ \mu\text{m/m}$$

The changes in the dimensions of the cross section are

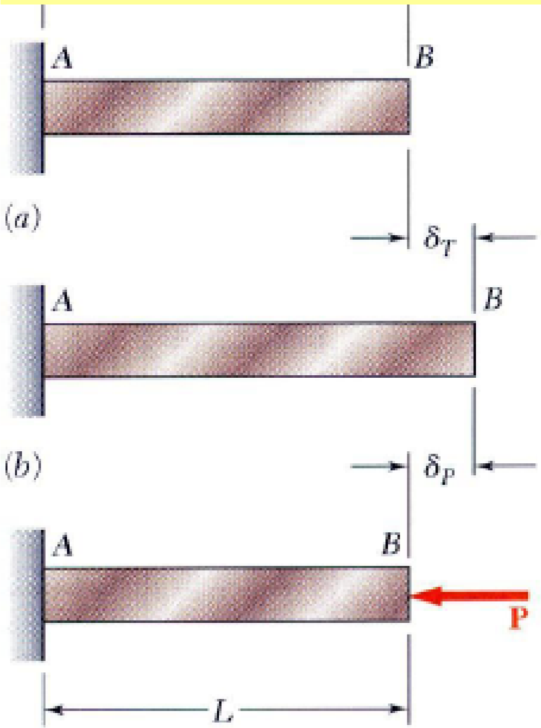
$$\delta_x = \varepsilon_x L_x = -\left[25.6\left(10^{-6}\right)(0.1)\right] = -2.56\,\mu\text{m} \text{ (Ans)}$$

$$\delta_y = \varepsilon_y L_y = -\left[25.6\left(10^{-6}\right)(0.05)\right] = -1.28\,\mu\text{m} \text{ (Ans)}$$

Thermal stress



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.



- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \quad \& \quad \delta_P = \frac{PL}{AE}$$

$\alpha =$ thermal expansion coef.

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0 \qquad \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Thermal stress

- Expansion or contraction of material is linearly related to temperature increase or decrease that occurs (for homogenous and isotropic material)
- From experiment, deformation of a member having length L is

$$\delta_T = \alpha \, \Delta T \, L$$

α = liner coefficient of thermal expansion. Unit measure strain per degree of temperature: $1/^{o}\mathrm{C}$ (Celsius) or $1/^{o}\mathrm{K}$ (Kelvin)

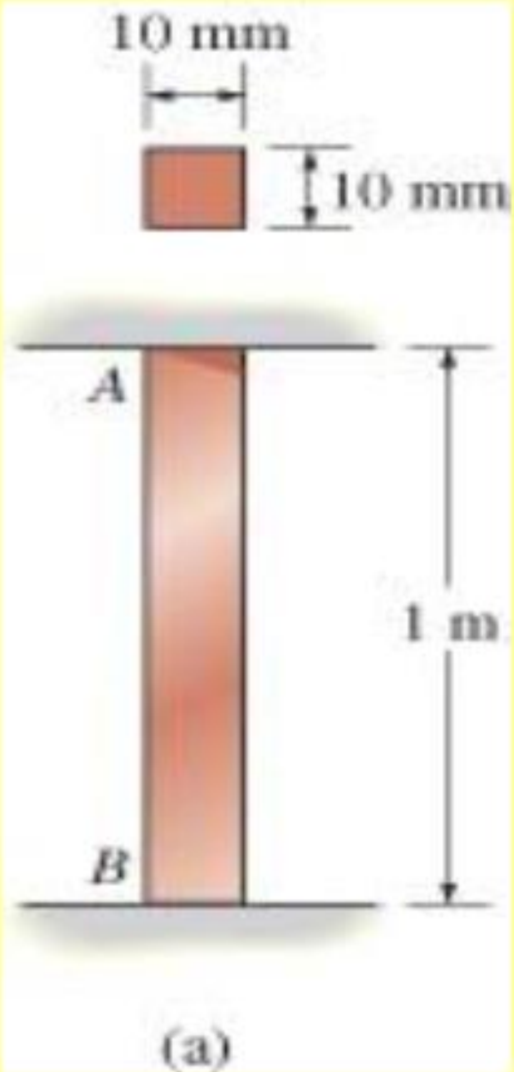
ΔT = algebraic change in temperature of member

δT = algebraic change in length of member

Thermal stress

A-36 steel bar shown is constrained to just fit between two fixed supports when $T_1 = 30^{\circ}C$.

If temperature is raised to $T_2 = 60^{\circ}C$, determine the average normal thermal stress developed in the bar.



Thermal stress

Equilibrium

As shown in free-body diagram,

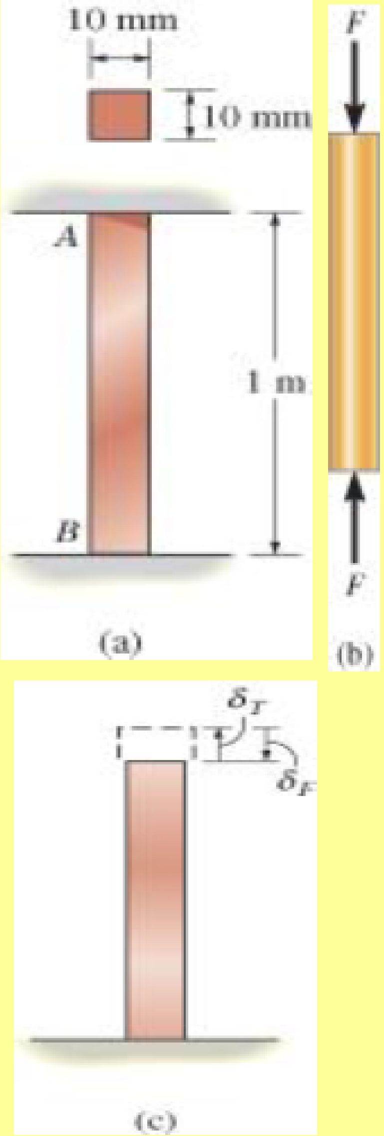
$$+\uparrow \sum F_y = 0; \qquad F_A = -F_B = F$$

Problem is statically indeterminate since the force cannot be determined from equilibrium.

Compatibility

Since $\delta_{A/B}$ =0, thermal displacement δ_T at A occur. Thus compatibility condition at A becomes:

$$+\uparrow \qquad \delta_{A/B} = 0 = \delta_T - \delta_F$$



Thermal stress

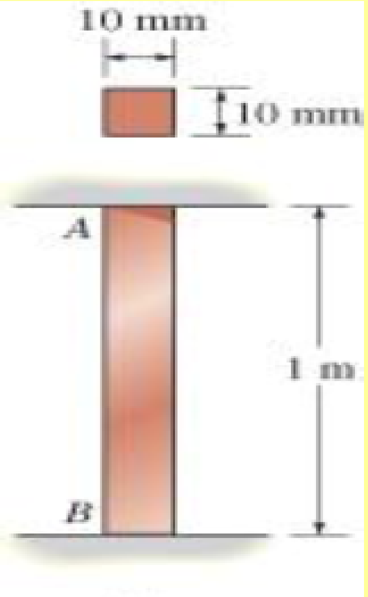
Compatibility

Apply thermal and load-displacement relationship,

$$\delta_{A/B} = 0 = \delta_T - \delta_F \Rightarrow\Rightarrow$$

$$0 = \alpha \Delta T L - \frac{FL}{AL}$$

$$F = \alpha \Delta T A E = \ldots = 7.2 \text{ kN}$$

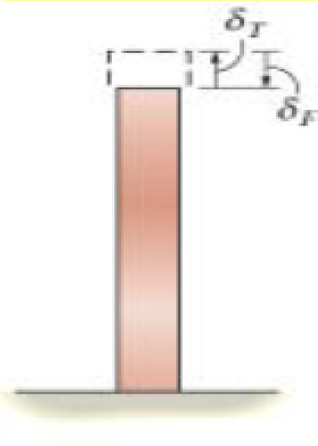


(a)

From magnitude of **F**, it's clear that changes in temperature causes large reaction forces in statically indeterminate members.

Average normal compressive stress is

$$\sigma = \frac{F}{A} = \ldots = 72 \text{ MPa}$$

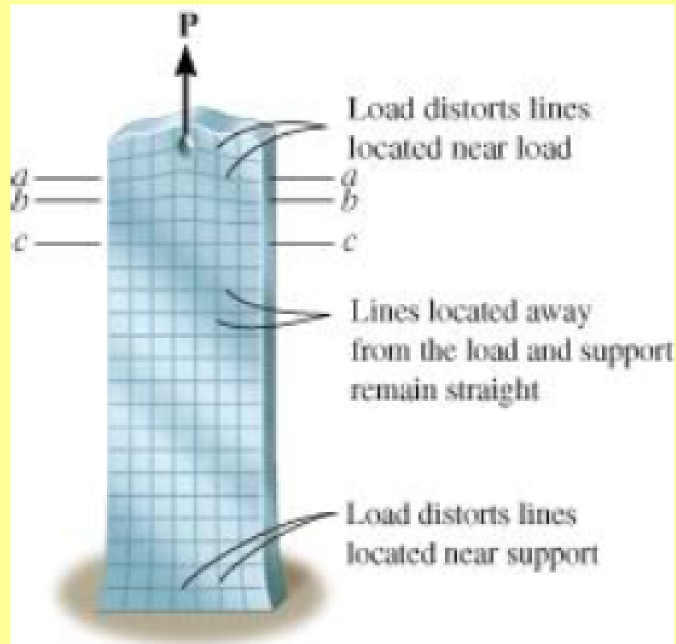


(c)

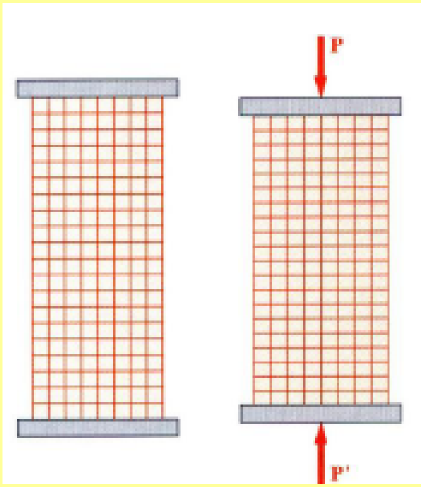
Stress localization

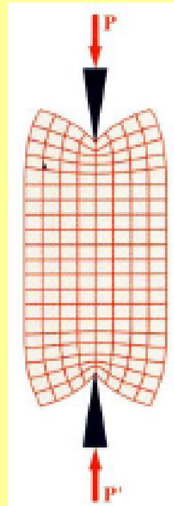
- Localized deformation occurs at each end, and the **deformations decrease as measurements are taken further away from the ends**
- **At section *c-c*, stress reaches almost uniform value as compared to *a-a*, *b-b*.**

- *Thus c-c* is sufficiently far enough away from **P** so the localized deformation "vanishes"

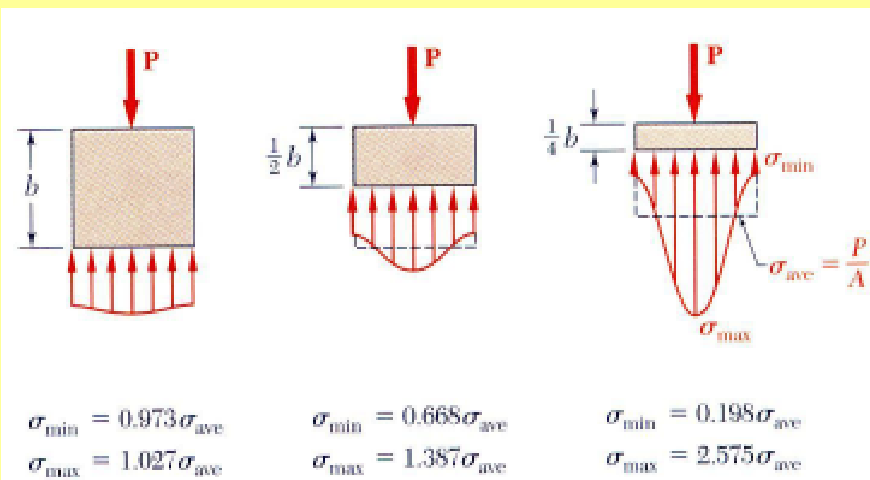


SAINT-VENANT'S PRINCIPLE



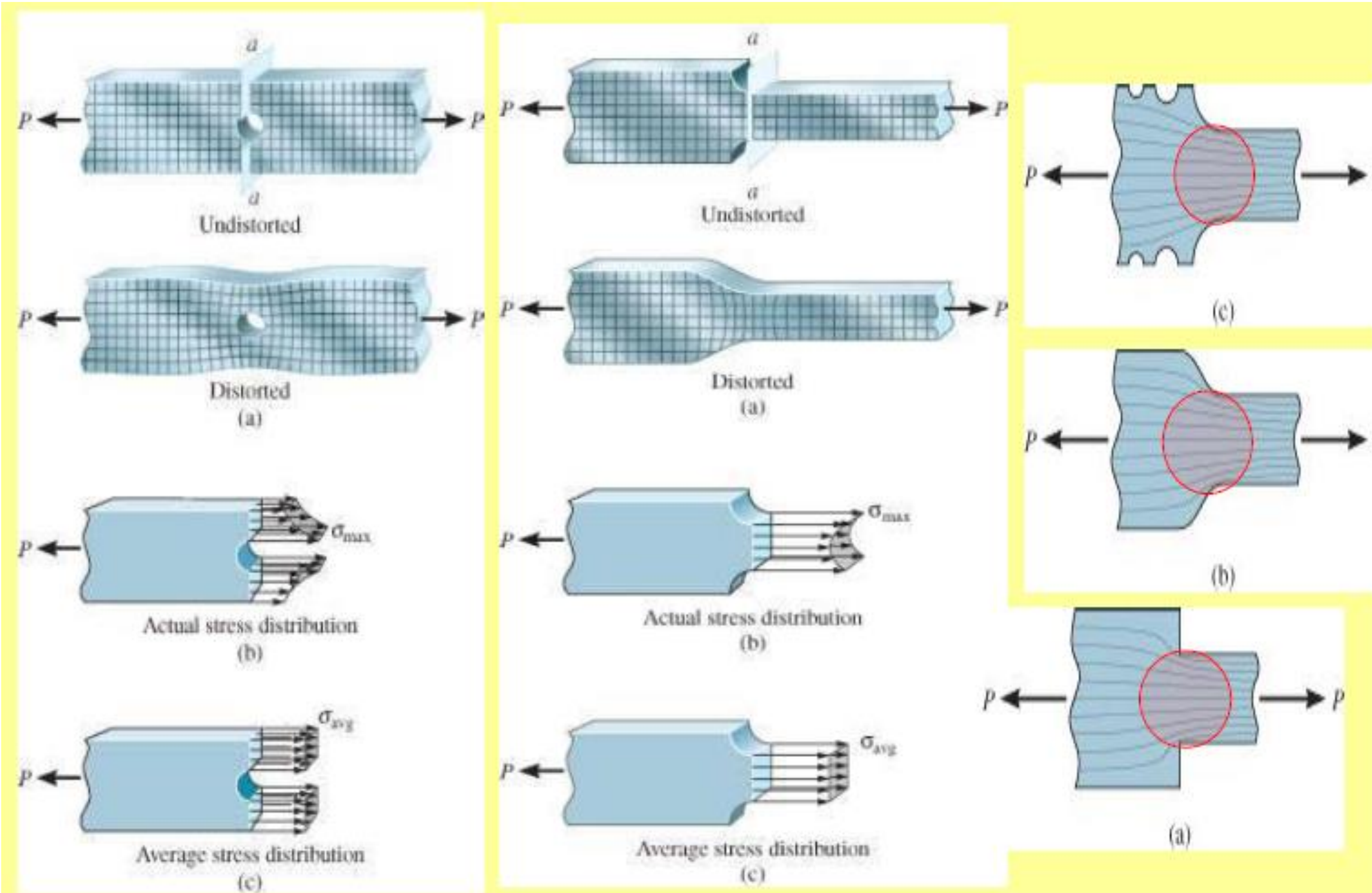


- Loads transmitted through rigid plates result in uniform distribution of stress and strain.
- Concentrated loads result in large stresses in the vicinity of the load application point.

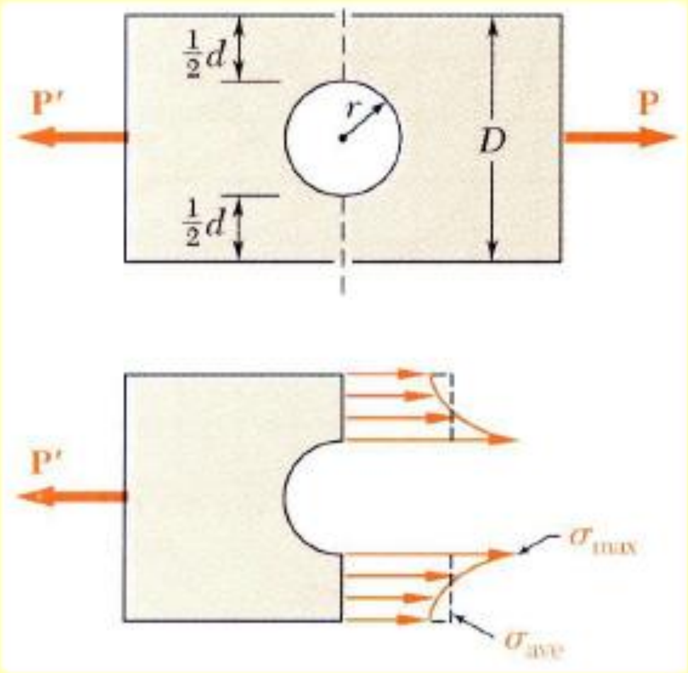


- Stress and strain distributions become uniform at a relatively short distance from the load application points.
- **Saint-Venant's Principle:**
  Stress distribution may be assumed independent of the mode of load application except in the immediate vicinity of load application points.

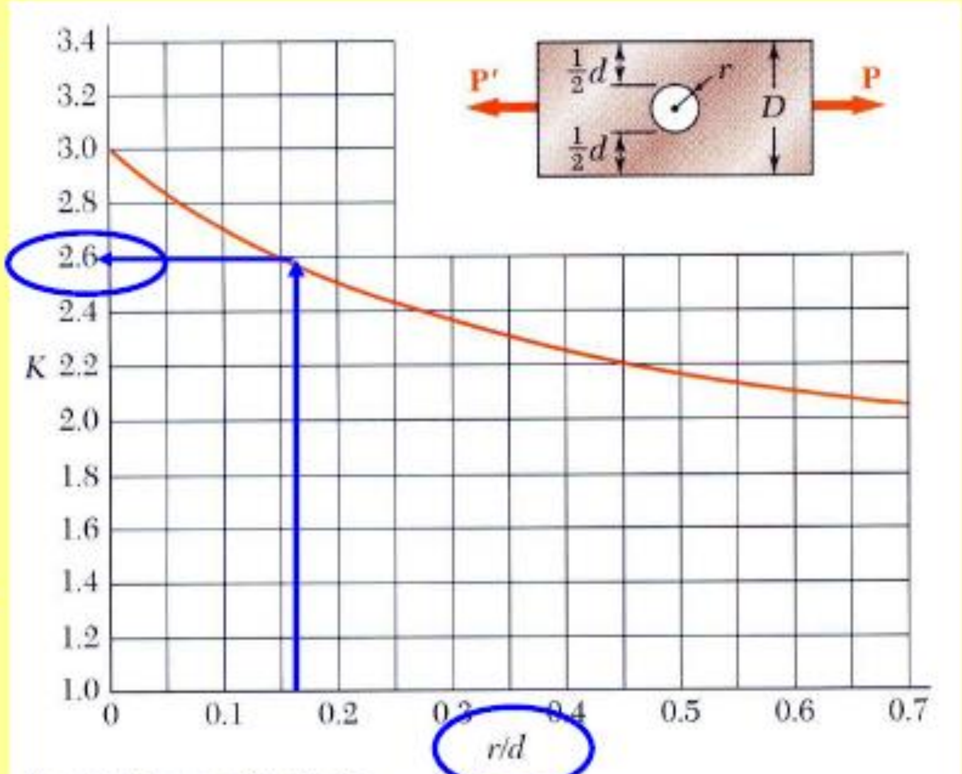
Stress concentration



Stress concentration



$$\sigma_{max} = K\sigma_{ave} = K\frac{P}{A_{net}}$$



(*a*) Flat bars with holes

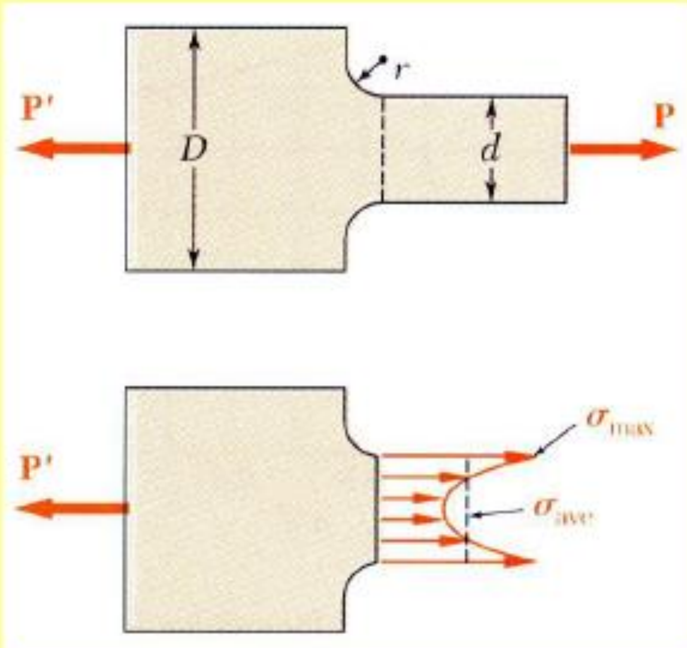
Discontinuities of cross section may result in high localized or *concentrated* stresses.

$$K = \frac{\sigma_{max}}{\sigma_{ave}} \quad \text{or} \quad \sigma_{ave} = \frac{\sigma_{max}}{K}$$

Stress concentration

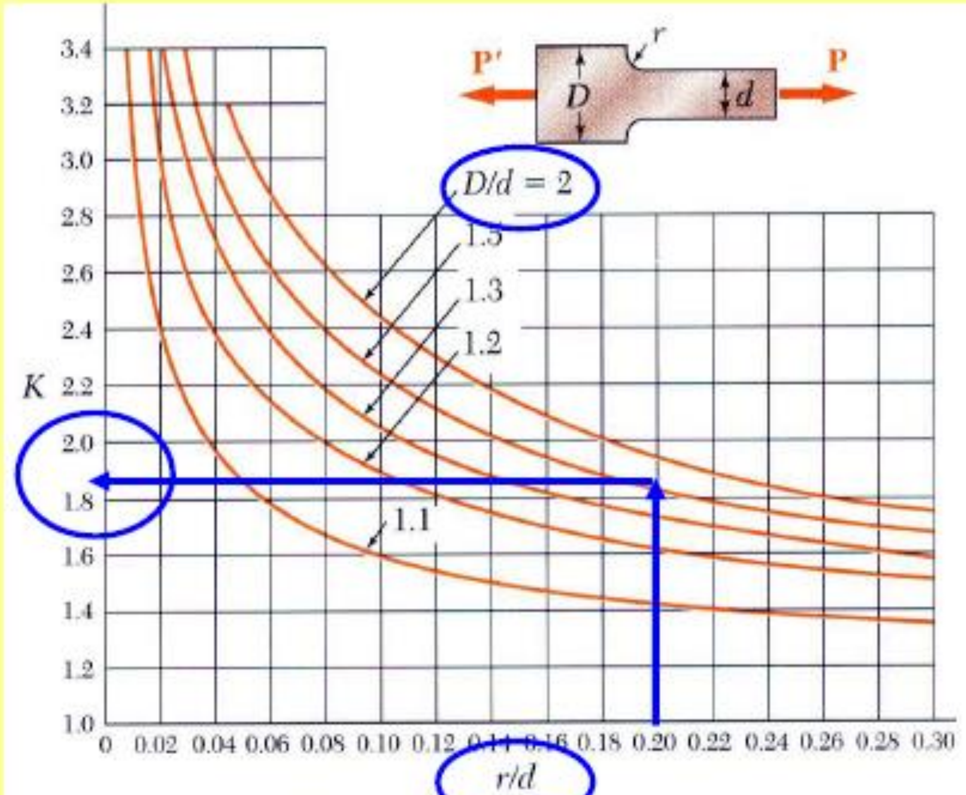
- *K* is independent of the bar's geometry and the type of discontinuity, only on the bar's geometry and the type of discontinuity.
- As size *r* of the discontinuity is decreased, stress concentration is increased.
- It is important to use stress-concentration factors in design when using brittle materials, but not necessary for ductile materials
- Stress concentrations also cause failure structural members or mechanical elements subjected to *fatigue loadings*

Stress concentration



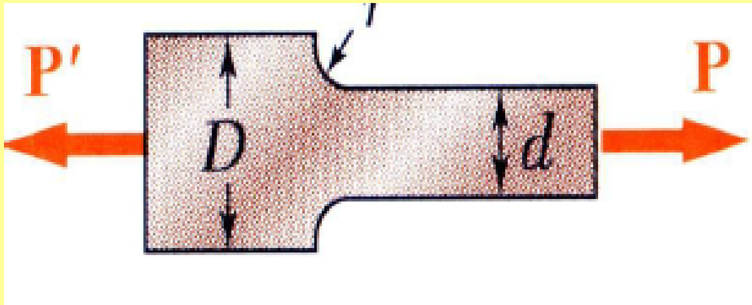
$$\sigma_{max} = K\sigma_{ave} = K\frac{P}{A_{net}}$$

$$K = \frac{\sigma_{max}}{\sigma_{ave}} \text{ or } \sigma_{ave} = \frac{\sigma_{max}}{K}$$



(b) Flat bars with fillets

Example



Determine the largest axial load P that can be safely supported by a flat steel bar consisting of two portions, both **10-mm-thick**, and respectively **40- and 60-mm-wide**, connected by fillets of **radius r = 8** mm. Assume an **allowable normal stress of 165 MPa.**

SOLUTION:

1). **Determine** **the geometric ratios** and **find the stress concentration factor** from Fig. 2.64*b*.

r/d, D/d

2). **Find the allowable average normal stress** using the material allowable normal stress and the stress concentration factor.

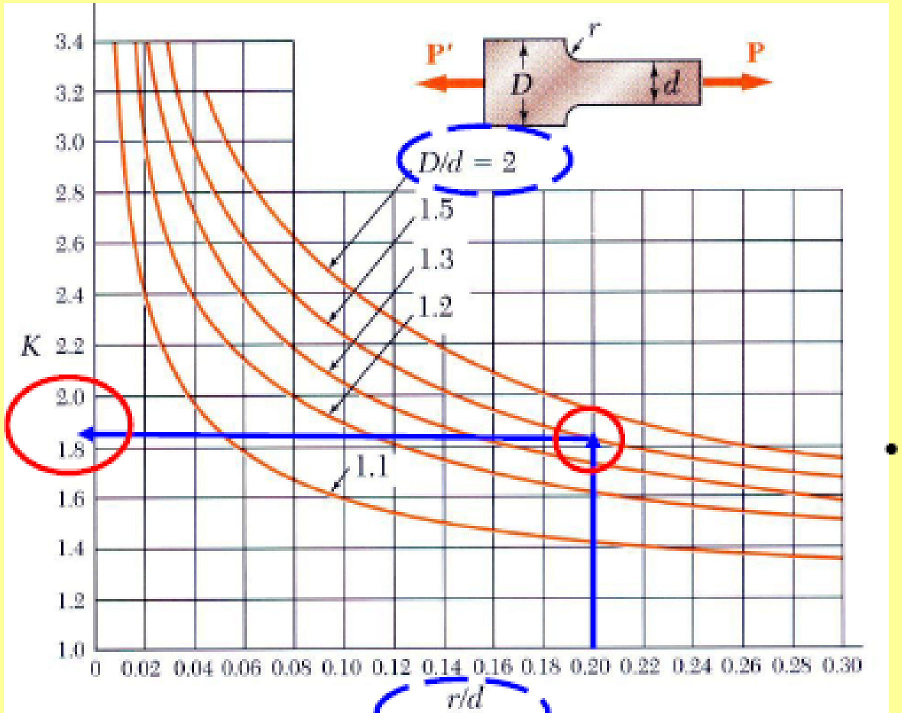
- Apply the definition of normal stress to find the allowable load.

Example

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.

$$\frac{D}{d}=\frac{60\,\text{mm}}{40\,\text{mm}}=1.50 \qquad \frac{r}{d}=\frac{8\,\text{mm}}{40\,\text{mm}}=0.20$$

$$K=1.82$$



(*b*) Flat bars with fillets

- Find the allowable average normal stress using the material allowable normal stress (165MPa) and the stress concentration factor (1.82).

$$\sigma_{ave}=\frac{\sigma_{max}}{K}=\frac{165\,\text{MPa}}{1.82}=90.7\,\text{MPa}$$

- Apply the definition of normal stress to find the allowable load.

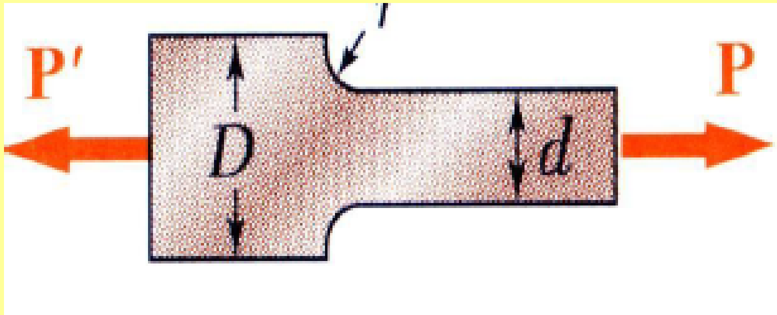
$$P=A\sigma_{ave}=(40\,\text{mm})(10\,\text{mm})(90.7\,\text{MPa})$$

$$=36.3\times10^{3}\,\text{N}$$

$$P=36.3\,\text{kN}$$

End

Example



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SOLUTION:

1). Determine the geometric ratios and **find the stress concentration factor** from Fig. 2.64*b*.

r/d, D/d

2). Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

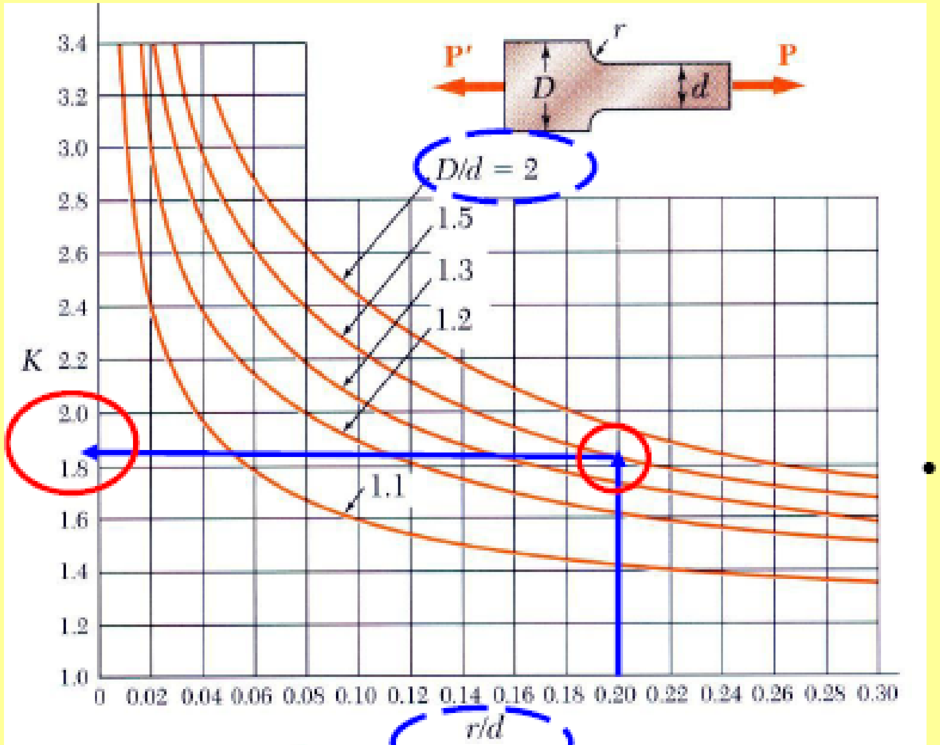
- Apply the definition of normal stress to find the allowable load.

Example

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.

$$\frac{D}{d} = \frac{60\,\text{mm}}{40\,\text{mm}} = 1.50 \qquad \frac{r}{d} = \frac{8\,\text{mm}}{40\,\text{mm}} = 0.20$$

$$K = 1.82$$



(*b*) Flat bars with fillets

- Find the allowable average normal stress using the material allowable normal stress (165MPa) and the stress concentration factor (1.82).

$$\sigma_{ave} = \frac{\sigma_{max}}{K} = \frac{165\,\text{MPa}}{1.82} = 90.7\,\text{MPa}$$

- Apply the definition of normal stress to find the allowable load.

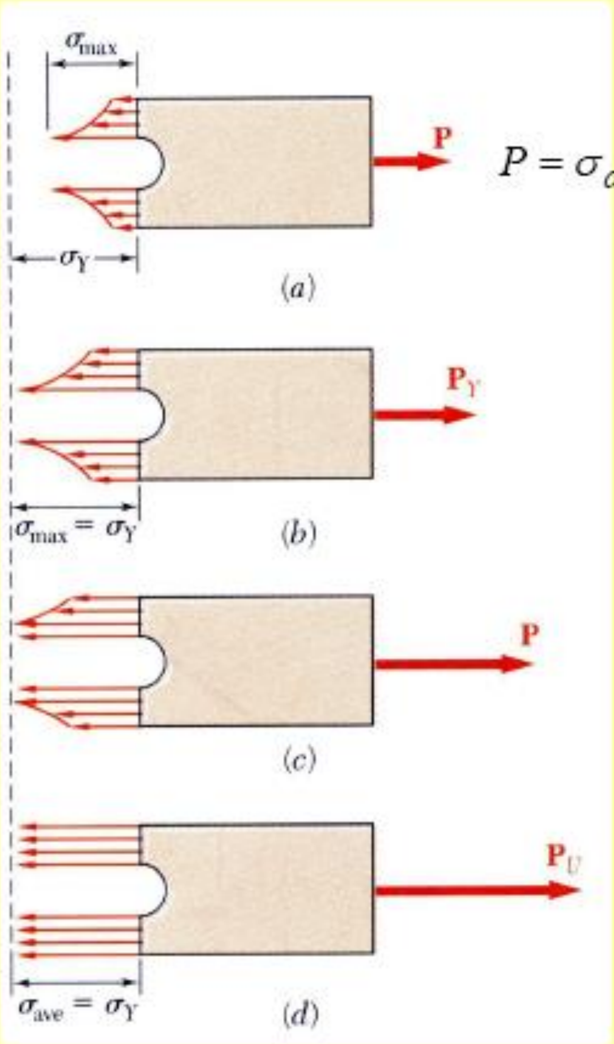
$$P = A\sigma_{ave} = (40\,\text{mm})(10\,\text{mm})(90.7\,\text{MPa})$$

$$= 36.3\times10^{3}\,\text{N}$$

$$P = 36.3\,\text{kN}$$

End

Stress distribution



$$P = \sigma_{ave} A = \frac{\sigma_{max} A}{K}$$

- Elastic deformation while maximum stress is less than yield stress.

$$P_Y = \frac{\sigma_Y A}{K}$$

- Maximum stress is equal to the yield stress at the maximum elastic loading.
- At loadings above the maximum elastic load, a region of plastic deformations develop near the hole.

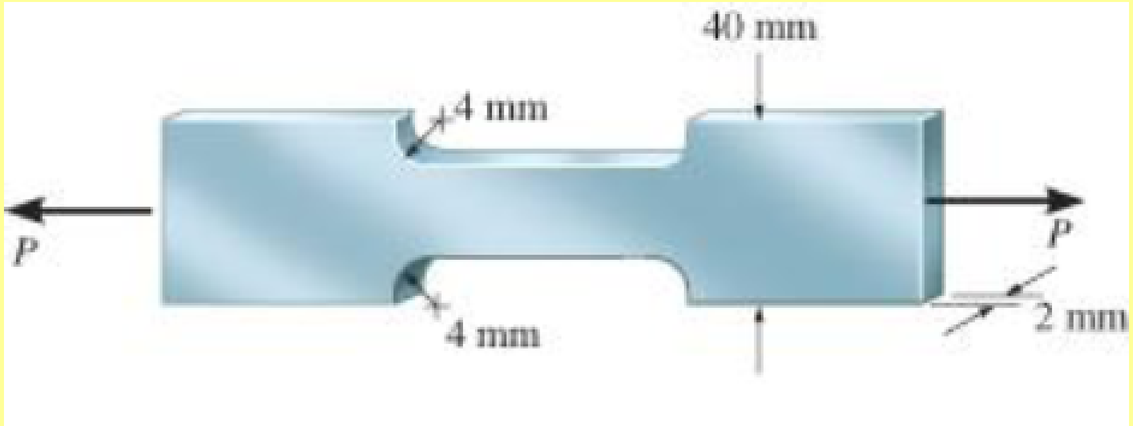
$$P_U = \sigma_Y A = K P_Y$$

- As the loading increases, the plastic region expands until the section is at a uniform stress equal to the yield stress.

Example

Steel bar shown assumed to be elastic perfectly plastic with $\sigma_Y = 250$ MPa.

Determine (a) maximum value of applied load **P** that can be applied without causing the steel to yield, (b) the maximum value of **P** that bar can support. Sketch the stress distribution at the critical section for each case.



(a)

Example

(a) When material behaves elastically, we must use a stress-concentration that is unique for the bar's geometry.

$$\frac{r}{n} = \frac{4\text{ mm}}{(40\text{ mm} - 8\text{ mm})} = 0.125$$

$$\frac{w}{h} = \frac{40\text{ mm}}{(40\text{ mm} - 8\text{ mm})} = 1.25$$

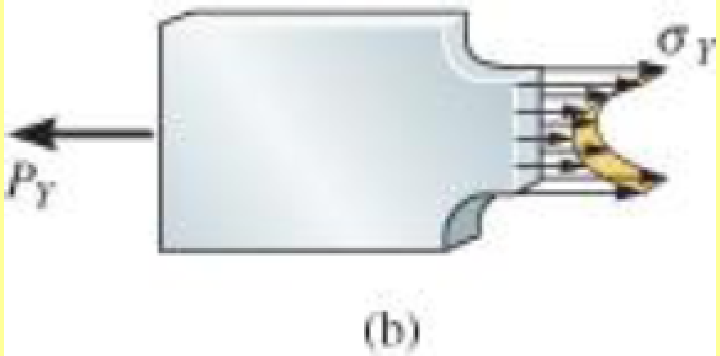
When $\sigma_{max} = \sigma_Y$. Average normal stress is $\sigma_{avg} = P/A$

$$\sigma_{max} = K\,\sigma_{avg}; \qquad \sigma_Y = K\left(\frac{P_Y}{A}\right)$$

$$\Rightarrow\Rightarrow P_Y = 16.0\text{ kN}$$

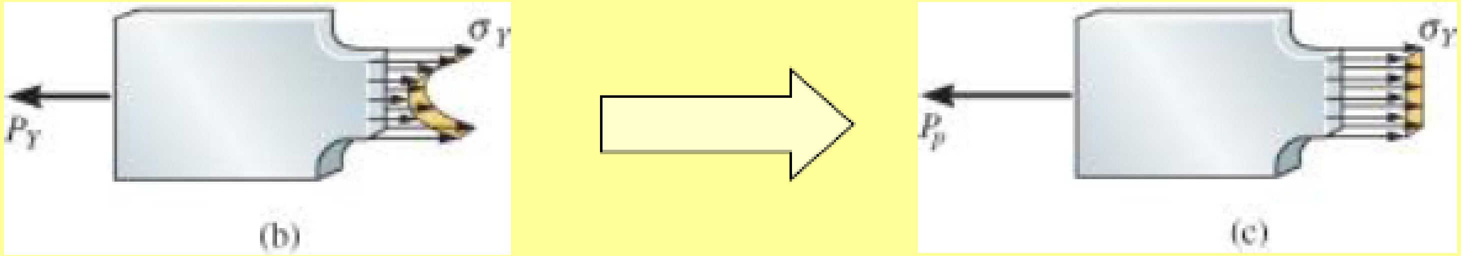
Example

(a) Load $\mathbf{P_Y}$ was calculated using the smallest x-section. Resulting stress distribution is shown. For equilibrium, the "volume" contained within this distribution must equal 9.14 kN.



Example

(b) Maximum load sustained by the bar causes all material at smallest x-section to yield. As **P** is increased to plastic load $\mathbf{P_P}$, the stress distribution changes as shown.



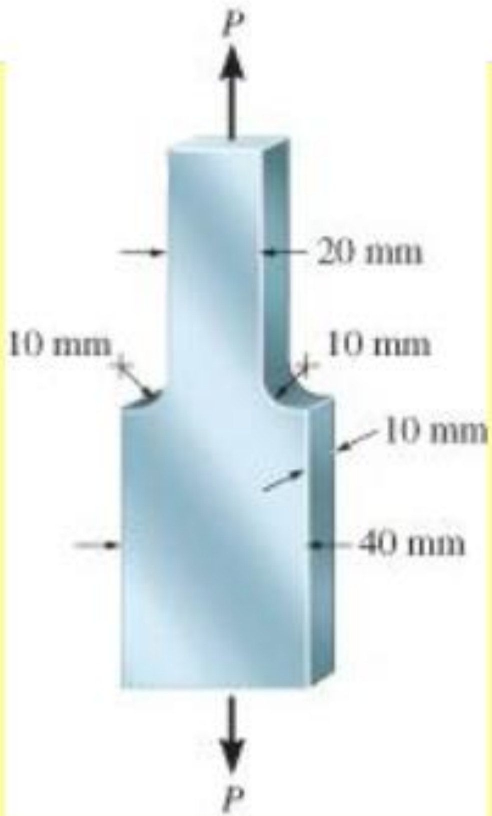
When $\sigma_{max} = \sigma_Y$. Average normal stress is $\sigma_{avg} = P/A$

$$\sigma_{max} = K\,\sigma_{avg}; \quad \sigma_Y = K\left(\frac{P_Y}{A}\right) \longrightarrow P_P = 16.0\ \text{kN}$$

Here, P_P equals the "volume" contained within the stress distribution, i.e., $P_P = \sigma_Y A$

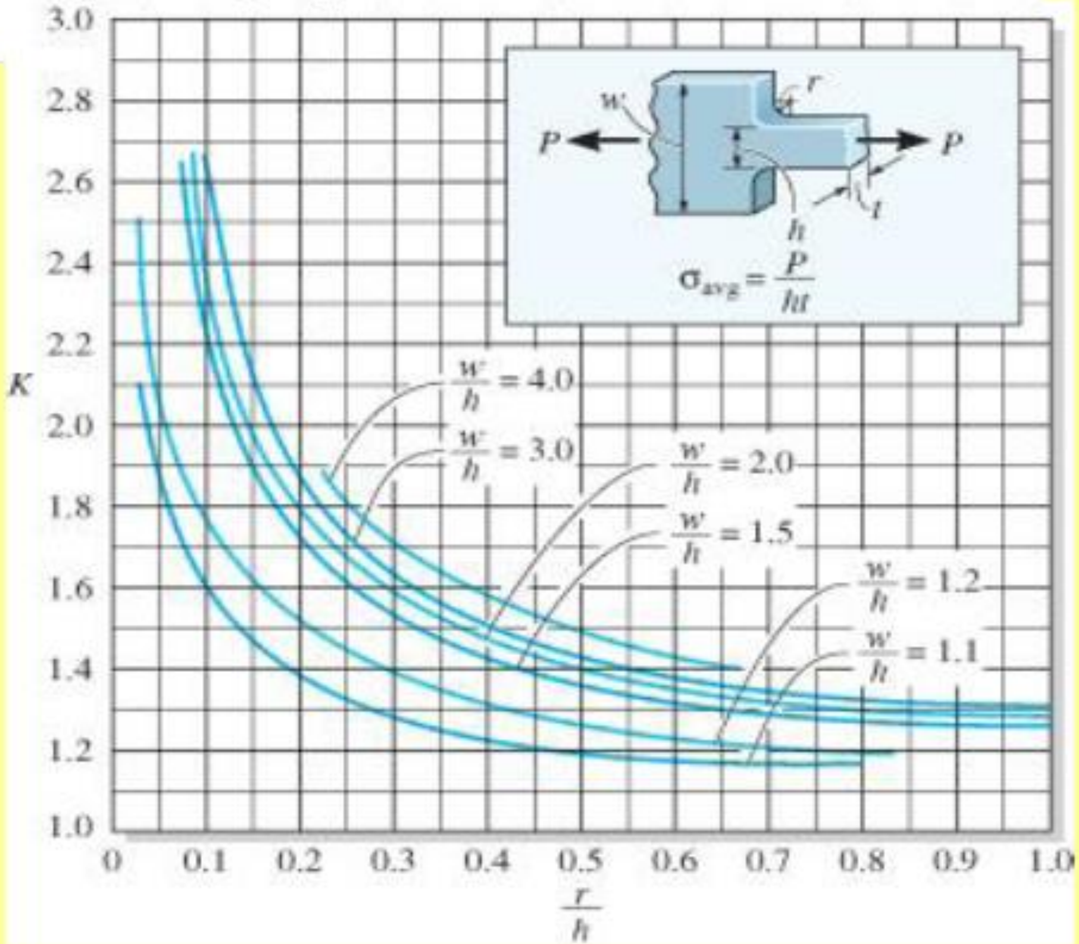
Example

Steel bar shown below has allowable stress, $\sigma_{allow} = 115$ MPa. Determine largest axial force **P** that the bar can carry.



Example

Because there is a shoulder fillet, stress-concentrating factor determined using the graph below



Example

Calculating the necessary geometric parameters yields

$$\frac{r}{n} = \frac{10\text{ mm}}{20\text{ mm}} = 0.50 \qquad \frac{w}{h} = \frac{40\text{ mm}}{20\text{ mm}} = 2$$

Thus, from the graph, $K = 1.4$

Average normal stress at *smallest* x-section,

$$\sigma_{avg} = \frac{P}{(20\text{ mm})(10\text{ mm})} = 0.005P\text{ N/mm}^2$$

Example

Applying Eqn 4-7 with $\sigma_{allow} = \sigma_{max}$ yields

$$\sigma_{allow} = K\,\sigma_{max}$$

$$115\ \mathrm{N/mm^2} = 1.4(0.005P)$$

$$P = 16.43(10^3)\ \mathrm{N} = 16.43\ \mathrm{kN}$$